Sampling of Continuous-Time Signals

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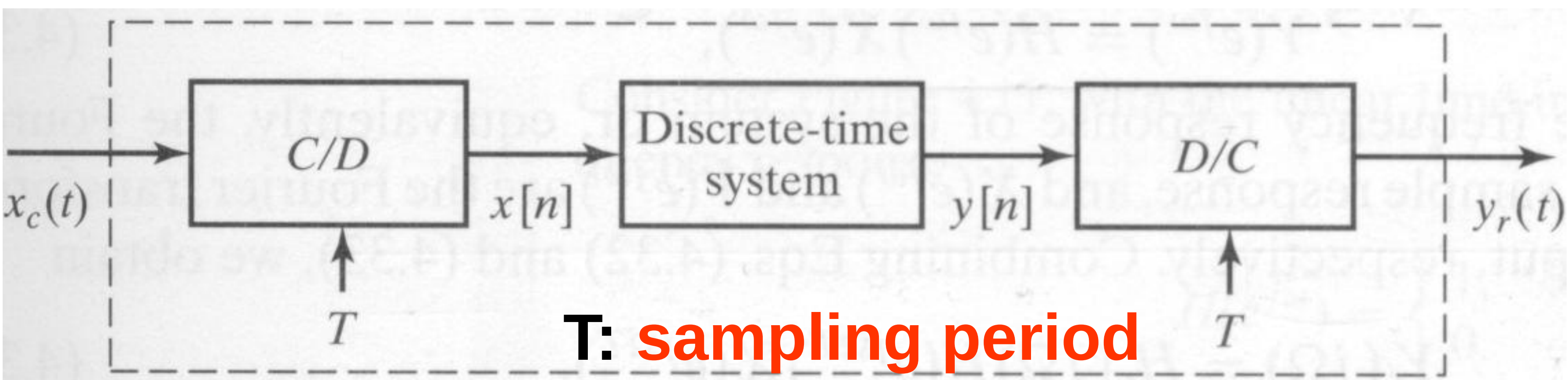
- Introduction
- Periodic Sampling
- Frequency-Domain Representation of Sampling
- Reconstruction of a Bandlimited Signal from its Samples
- Discrete-Time Processing of Continuous-Time signals

Introduction

- Continuous-time signal processing can be implemented through a process of sampling, discrete-time processing, and the subsequent reconstruction of a continuous-time signal.



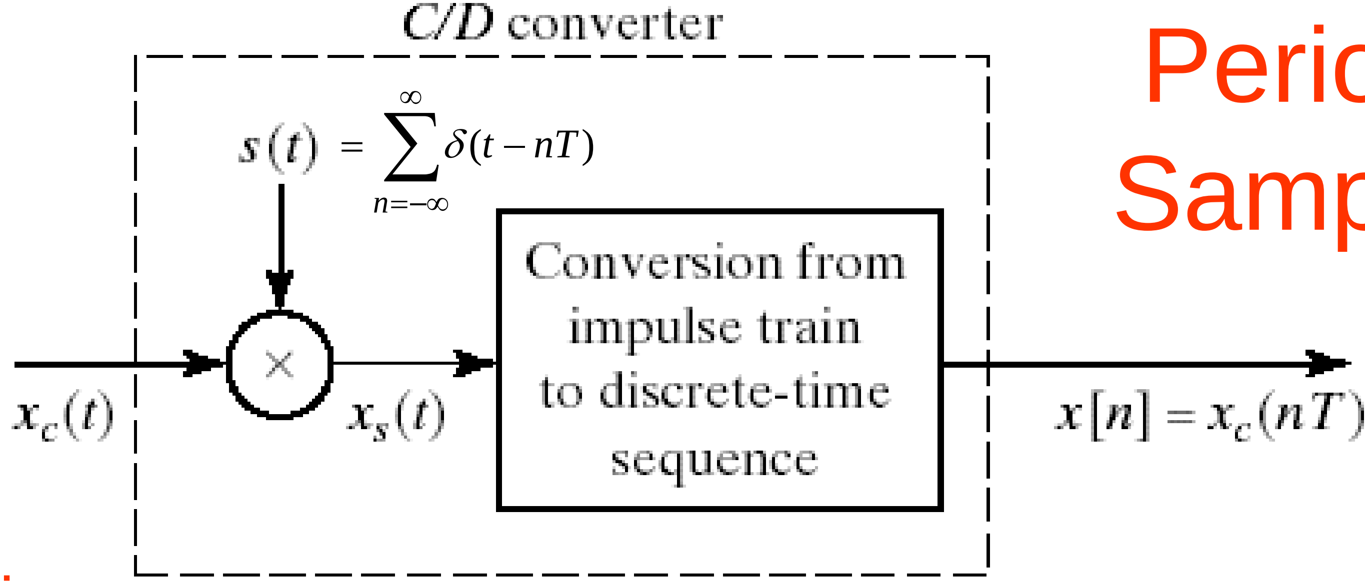
T: sampling period

$$x[n] = x_c(nT), \quad -\infty < n < \infty$$

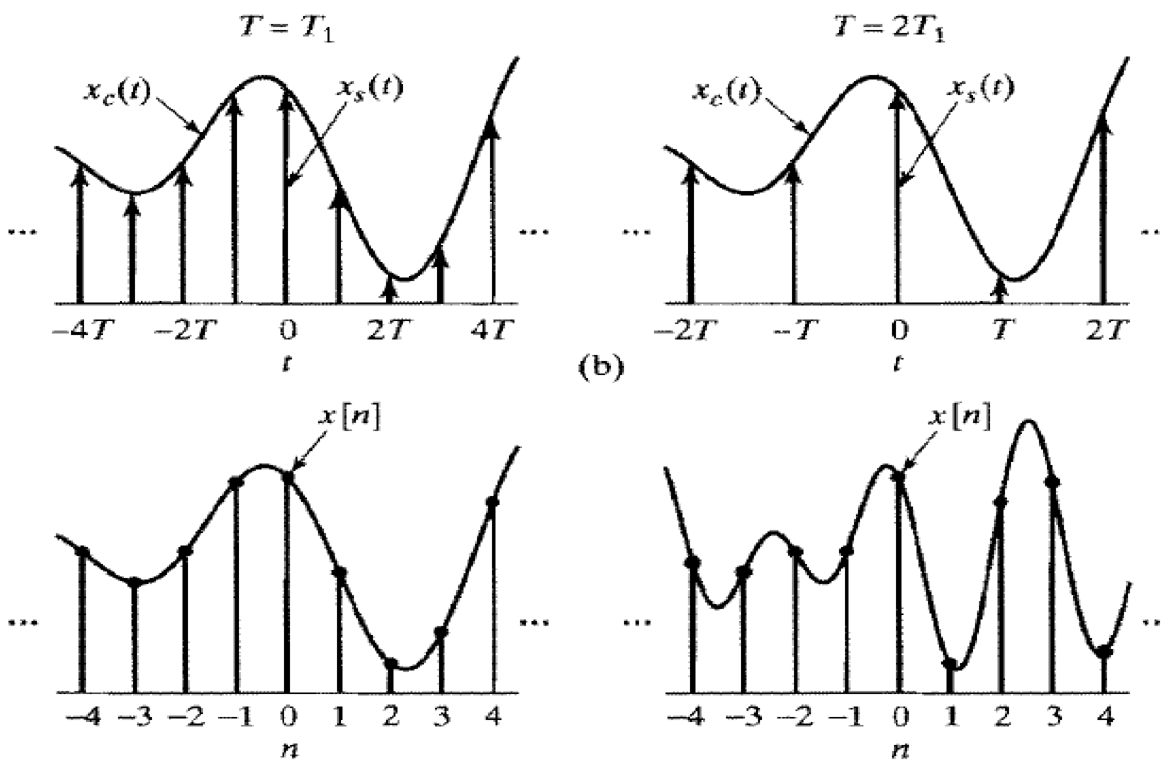
f=1/T: sampling frequency

$$\Omega_s = 2\pi/T, \quad (rad/s)$$

Periodic Sampling



Continuous-time signal



T: sampling period

Frequency-Domain Representation of Sampling

$$s(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

T: sample period; fs=1/T:sample rate

Ωs=2π/T: sample rate

$$x_s(t)=x_c(t)s(t)=x_c(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)=\sum_{n=-\infty}^{\infty}x_c(nT)\delta(t-nT)$$

$$x[n]=x_c(t)\big|_{t=nT}=x_c(nT)$$

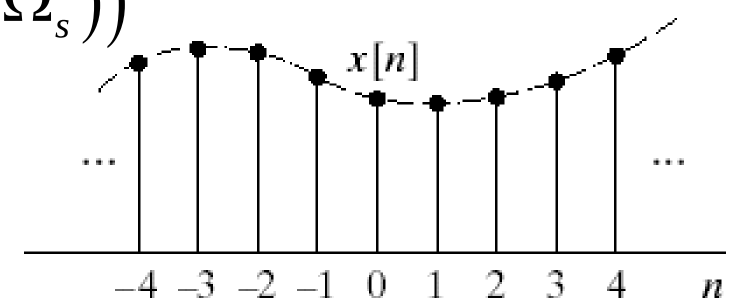
$$S(j\Omega)=\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\Omega-k\Omega_s)$$

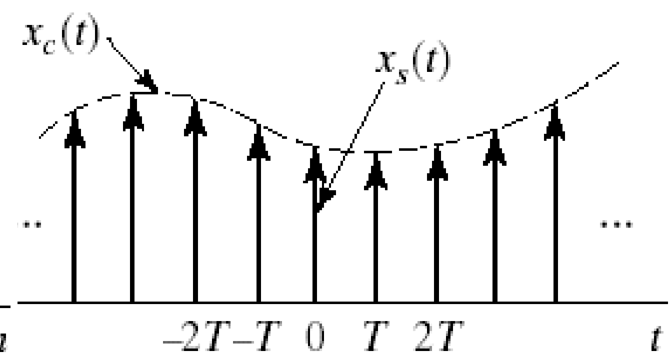
$$X_s(j\Omega)=\frac{1}{2\pi}X_c(j\Omega)*S(j\Omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S(j\theta)X_c(j(\Omega-\theta))d\theta$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\theta-k\Omega_s)X_c(j(\Omega-\theta))d\theta=\frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(\theta-k\Omega_s)X_c(j(\Omega-\theta))d\theta$$

$$=\frac{1}{T}\sum_{k=-\infty}^{\infty}X_c(j(\Omega-k\Omega_s))$$

Representation of $X_s(j\Omega)$ in terms of $X(e^{jw})$



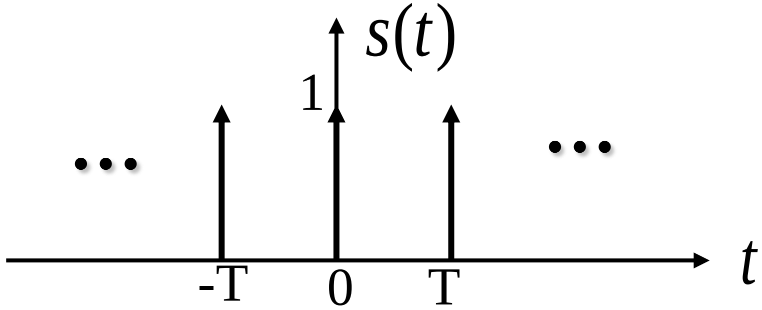


$$S(j\Omega)=\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\Omega-k\Omega_s)$$

T： sample period; fs=1/T:sample rate;Ωs=2π/T:sample rate

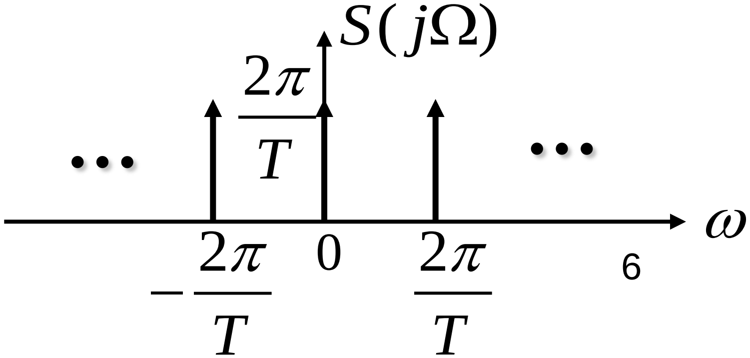
$$s(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)=\sum_{n=-\infty}^{\infty}a_k e^{jk\Omega_s t}=\frac{1}{T}\sum_{n=-\infty}^{\infty}e^{jk\Omega_s t}$$

$$a_k=\frac{1}{T}\int_{-T/2}^{+T/2}\delta(t)e^{-jk\Omega_s t}dt=\frac{1}{T}$$



$$e^{jk\Omega_s t}\overset{F}{\longleftrightarrow}2\pi\delta(\Omega-k\Omega_s)$$

$$S(j\Omega)=\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\Omega-k\Omega_s)$$



Representation of $X\left(e^{jw}\right)$ in terms of $X_s\left(j\Omega\right)$, $X_c\left(j\Omega\right)$

$$x_s(t) = x_c(t)s(t) = x_c(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) = \sum_{n=-\infty}^{\infty}x_c(nT)\delta(t-nT)$$

$$X_s(j\Omega) = \int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}x_c(nT)\delta(t-nT)e^{-j\Omega t}dt$$

$$= \sum_{k=-\infty}^{\infty}x_c(nT)e^{-j\Omega Tn}$$

$$x[n] = x_c(nT)$$

$$\Omega T = \omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty}x_c(nT)e^{-j\omega n} = X(e^{j\Omega T})$$

DTFT

$$= X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty}X_c\left(j\left(\Omega - k\Omega_s\right)\right)$$

$$\Omega_s = \frac{2\pi}{T},$$

Representation of $X\left(e^{jw}\right)$ in terms of $X_s\left(j\Omega\right)$, $X_c\left(j\Omega\right)$

$$X(e^{j\omega}) = X(e^{j\Omega T}) = X_s\left(j\Omega\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - k\Omega_s\right)\right)$$

DTFT **Continuous FT**

$$\Omega = \omega / T$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

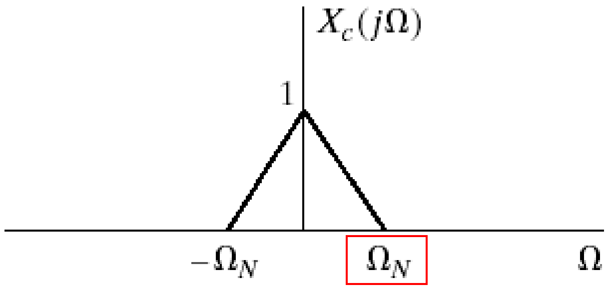
$$if \;\; X_c\left(j\Omega\right) = 0, \quad \Omega \geq \frac{\pi}{T}$$

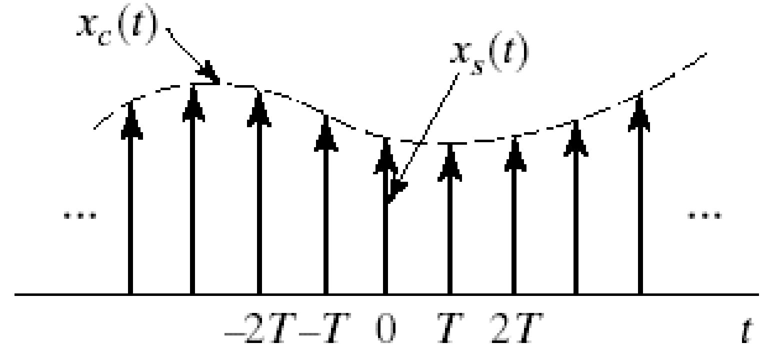
$$then \; X(e^{j\omega}) = \frac{1}{T} X_c\left(j\frac{\omega}{T}\right) \qquad \left|\omega\right| < \pi$$

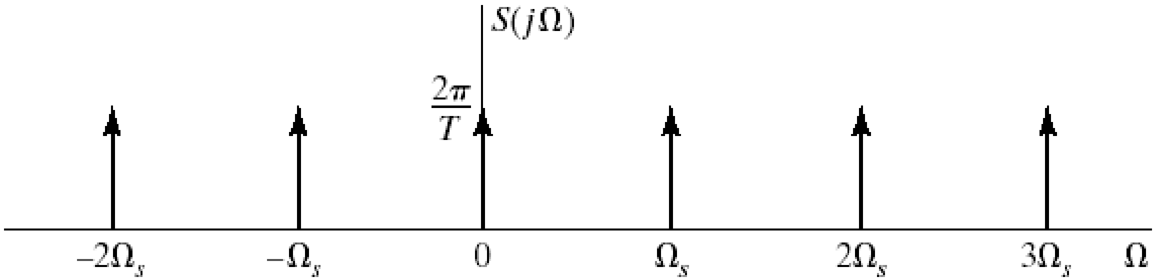
Nyquist Sampling Theorem

- Let $x_c(t)$ be a bandlimited signal with $X_c(j\Omega)=0 \quad for\ |\Omega|\geq\Omega_N$. Then $x_c(t)$ is uniquely determined by its samples $x[n]=x_c(nT), \quad n=0,\pm1,\pm2,\ldots$,if $\Omega_s=\frac{2\pi}{T}\geq 2\Omega_N$

- The frequency Ω_N is commonly referred as the *Nyquist frequency*.
- The frequency $2\Omega_N$ is called the *Nyquist rate*.

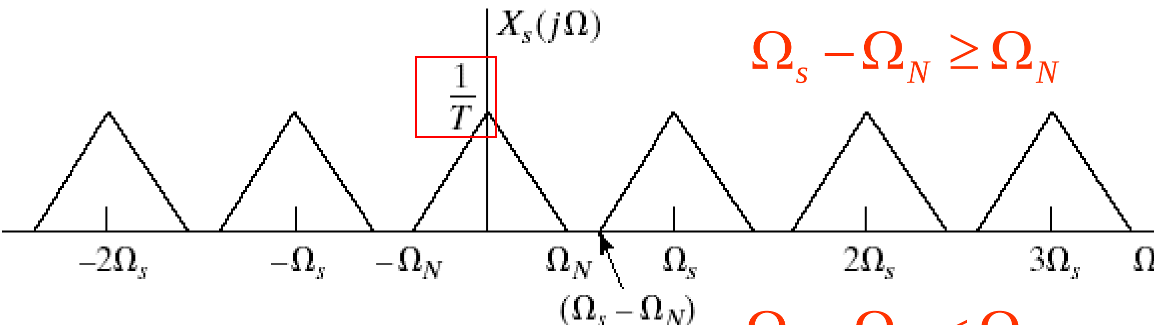






frequency spectrum of ideal sample signal

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$



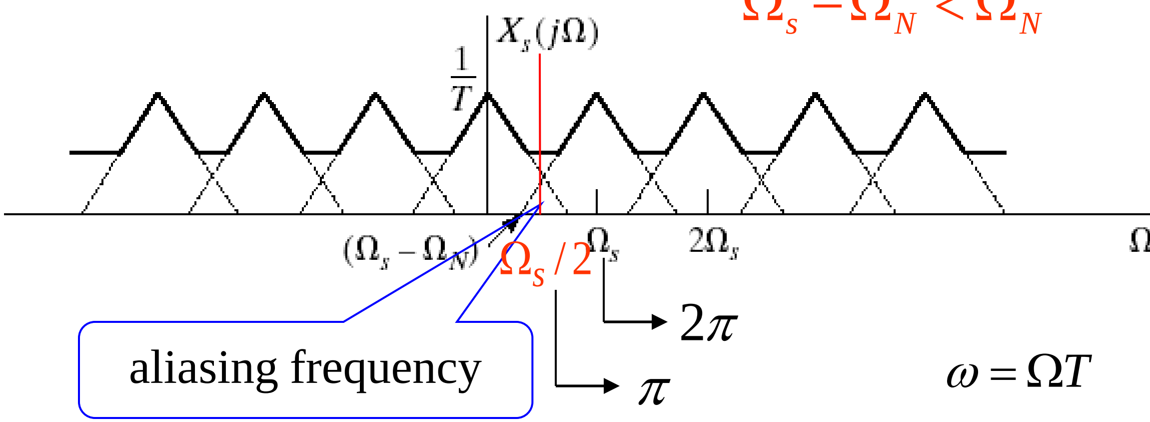
$\Omega_s - \Omega_N \geq \Omega_N$

No aliasing

$$X(e^{j\omega}) = X_s(j\Omega)\big|_{\Omega=\omega/T}$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\omega - k2\pi)/T)$$

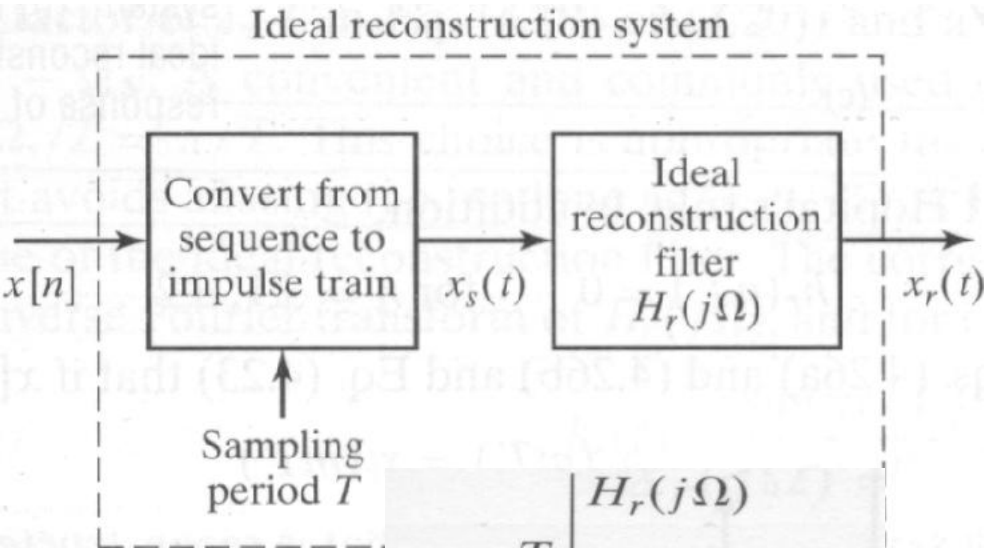
$\Omega_s - \Omega_N < \Omega_N$

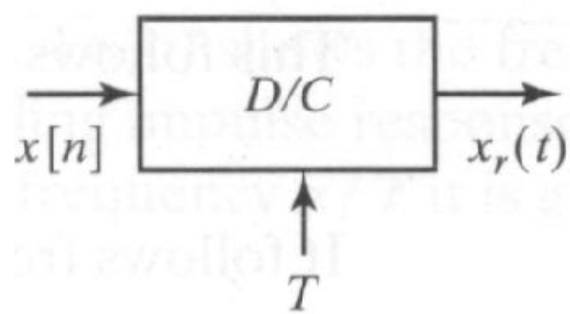


$\omega = \Omega T$

aliasing

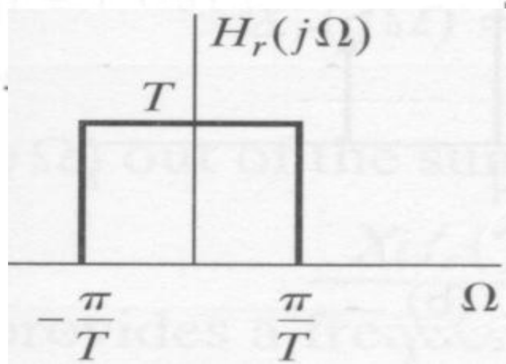
Reconstruction of a Bandlimited Signal from its Samples





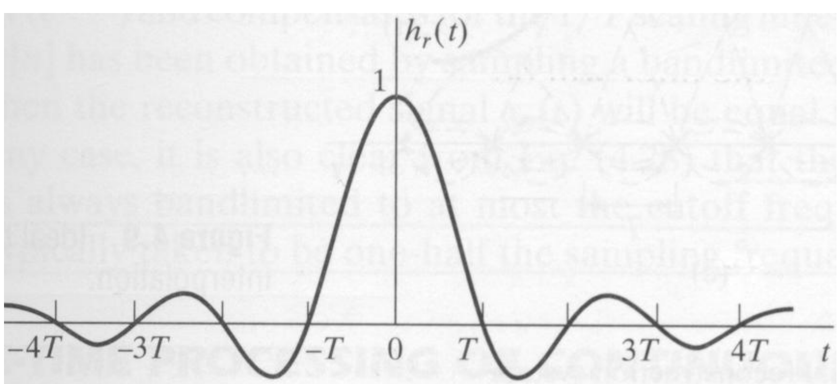
$$h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T}$$

Gain: T



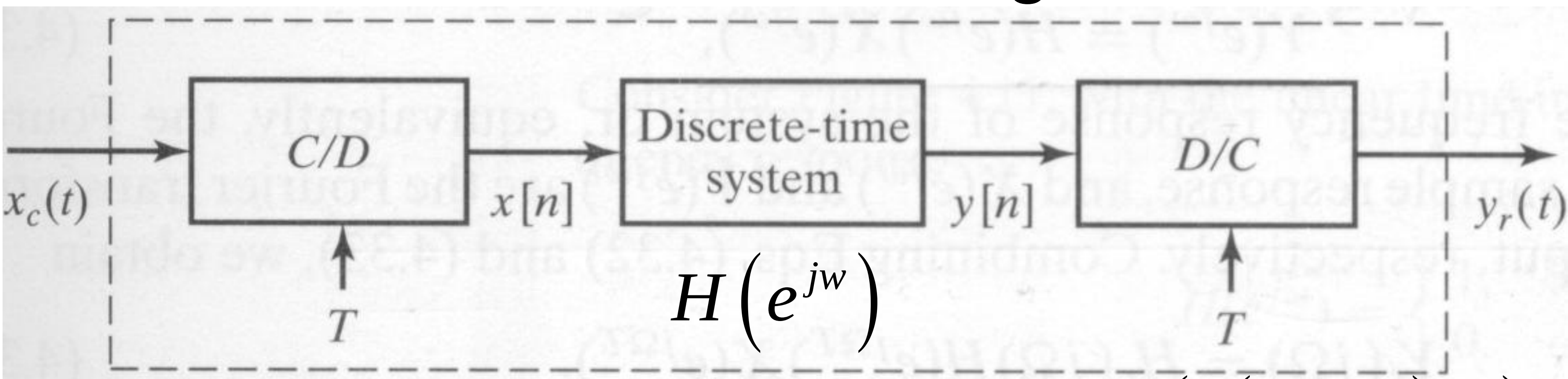
$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h_r(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} x[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}$$



$$X_r(j\Omega) = H_r(j\Omega) X\left(e^{j\Omega T}\right)$$

Discrete-Time Processing of Continuous-Time signals



$$x[n] = x_c(nT)$$

$$y_r(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)/T}$$

$$X\left(e^{jw}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{w}{T} - \frac{2\pi k}{T}\right)\right)$$

$$Y_r(j\Omega) = H_r(j\Omega) Y\left(e^{j\Omega T}\right) = \begin{cases} TY\left(e^{j\Omega T}\right), & |\Omega| < \frac{\pi}{T} \\ 0, & otherwise \end{cases}$$

C/D Converter

- Output of C/D Converter

$$x[n] = x_c(nT)$$

$$X\left(e^{jw}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{w}{T} - \frac{2\pi k}{T}\right)\right)$$

D/C Converter

- Output of D/C Converter

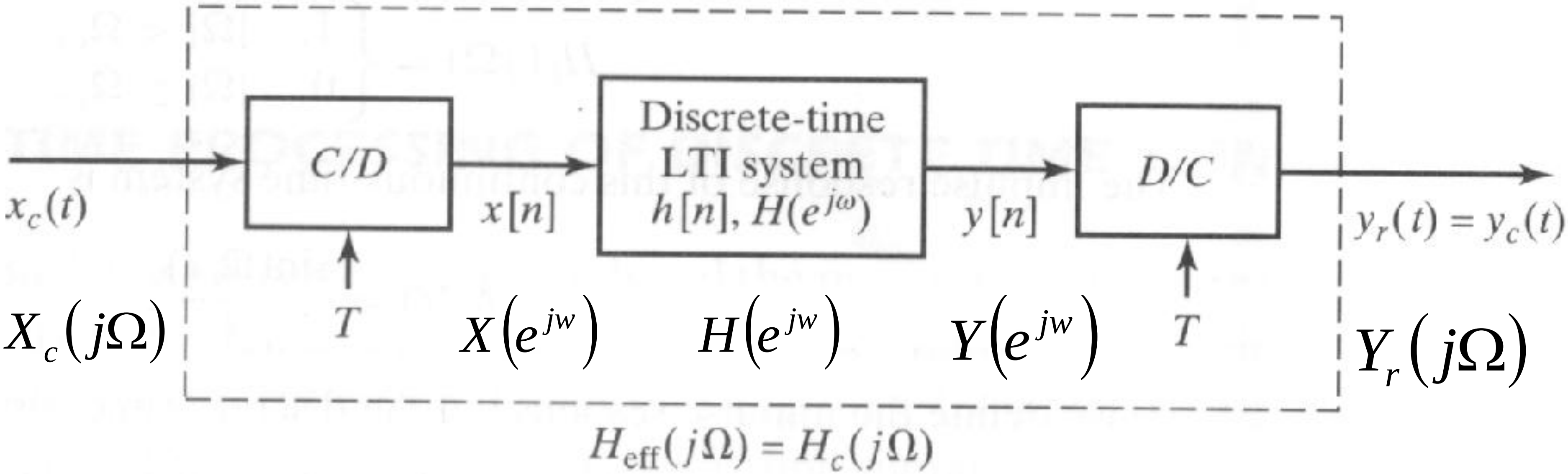
$$y_r(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)/T}$$

$$Y_r(j\Omega) = H_r(j\Omega) Y(e^{j\Omega T})$$

$$= \begin{cases} TY(e^{j\Omega T}), & |\Omega| < \frac{\pi}{T} \\ 0, & otherwise \end{cases}$$

$$H_r(j\Omega) = \begin{cases} T, & |\Omega| < \frac{\pi}{T} \\ 0, & otherwise \end{cases}$$

Linear Time-Invariant Discrete-Time Systems



$X_c(j\Omega)$ $X(e^{jw})$ $H(e^{jw})$ $Y(e^{jw})$ $Y_r(j\Omega)$

$$Y(e^{jw}) = H(e^{jw})X(e^{jw})$$

$$Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})X(e^{j\Omega T})$$

$$= H_r(j\Omega)H(e^{j\Omega T})\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right) = \begin{cases} H(e^{j\Omega T})X_c(j\Omega), & |\Omega| < \frac{\pi}{T} \\ 0, & |\Omega| \geq \frac{\pi}{T} \end{cases}$$

Linear and Time-Invariant

- Linear and time-invariant system behavior depends on two factors:
- First, the discrete-time system must be linear and time invariant.
- Second, the input signal must be bandlimited, and the sampling rate must be high enough to satisfy Nyquist Sampling Theorem.

$$Y_r(j\Omega) = H_r(j\Omega)H\left(e^{j\Omega T}\right)X\left(e^{j\Omega T}\right)$$

$$= H_r(j\Omega)H\left(e^{j\Omega T}\right)\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)$$

$$If \;\; X_c(j\Omega) = 0 \;\; for \; |\Omega| \geq \pi/T, \qquad H_r(j\Omega) = \begin{cases} T, & |\Omega| < \dfrac{\pi}{T} \\ 0, & otherwise \end{cases}$$

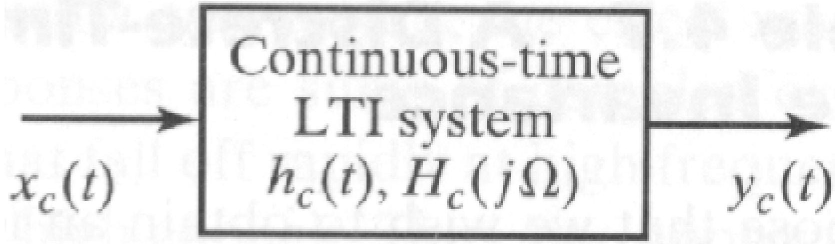
$$Y_r(j\Omega) = \begin{cases} H\left(e^{j\Omega T}\right)X_c(j\Omega), & |\Omega| < \dfrac{\pi}{T} \\ 0, & |\Omega| \geq \dfrac{\pi}{T} \end{cases}$$

$$Y_r(j\Omega) = H_{eff}(j\Omega)X_c(j\Omega)$$

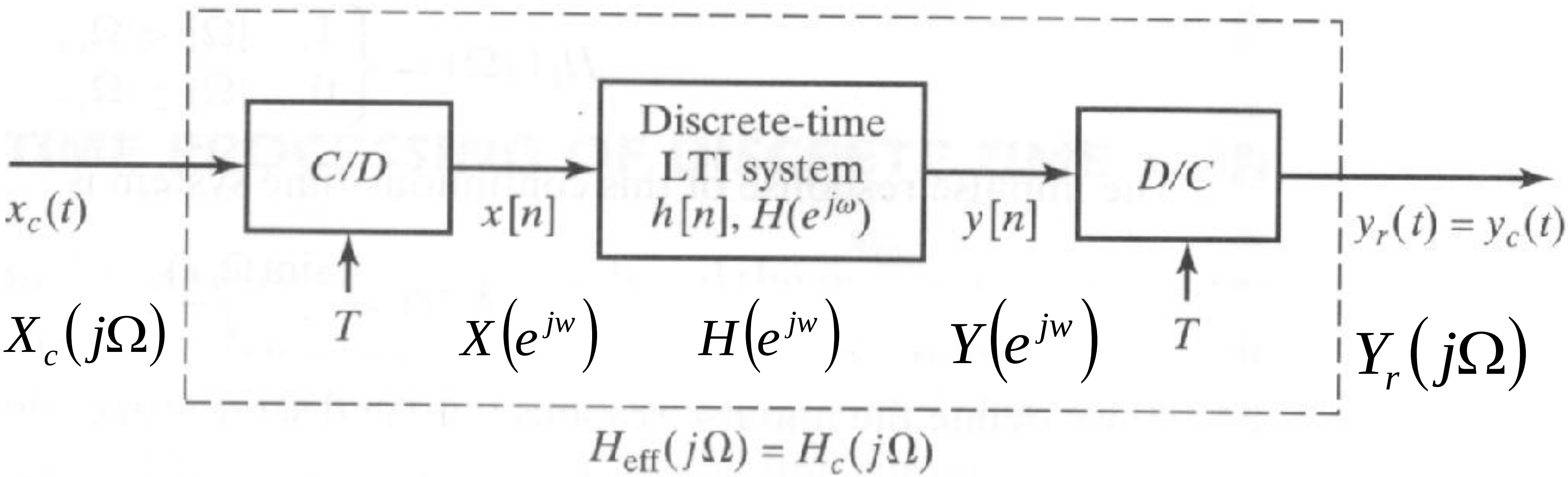
$$H_{eff}(j\Omega) = \begin{cases} H\left(e^{j\Omega T}\right), & |\Omega| < \dfrac{\pi}{T} \\ 0, & |\Omega| \geq \dfrac{\pi}{T} \end{cases}$$

Impulse Invariance

Given:



Design: $H\left(e^{jw}\right) \Leftarrow H_c\left(j\Omega\right),\qquad h[n] \Leftarrow h_c\left(nT\right)$



$$h[n] = T h_c\left(nT\right)$$

$$H_c\left(j\Omega\right) = H_{eff}\left(j\Omega\right) = \begin{cases} H\left(e^{j\Omega T}\right), & |\Omega| < \dfrac{\pi}{T} \\ 0, & |\Omega| \geq \dfrac{\pi}{T} \end{cases}$$

impulse-invariant version of the continuous-time system

Impulse Invariance

- Two constraints

1. $$H\left(e^{j\omega}\right) = H_c\left(j\omega/T\right), \quad |\omega| < \pi$$
2. T *is chosen such that* $\Omega_C < \pi / T$

$$H_c\left(j\Omega\right) = 0, \quad |\Omega| \ge \pi/T$$

$$h[n] = T h_c(nT)$$

The *discrete-time system* *is called* *an impulse-invariant* version *of the* continuous-time system

$$h[n] = h_c(nT) \Longrightarrow X(e^{j\omega}) = \frac{1}{T} X_c\left(j\frac{\omega}{T}\right)$$

$$h[n] = T h_c(nT) \Longleftarrow X(e^{j\omega}) = X_c\left(j\frac{\omega}{T}\right) \qquad |\omega| < \pi$$